

# THE ANDREWS-OLSSON IDENTITY AND BESSENRODT INSERTION ALGORITHM ON YOUNG WALLS.

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**ABSTRACT.** We extend the Andrews-Olsson identity to the setting of two-colored partition. Regarding the sets of reduced Young walls of type  $A_{2n}^{(2)}$ ,  $A_{2n-1}^{(2)}$ ,  $B_n^{(1)}$ ,  $D_n^{(1)}$  and  $D_{n+1}^{(2)}$  as the sets of two-colored partitions, the extended Andrews-Olsson identity implies that the generating functions of the sets of reduced Young walls have very simple formulae:

$$\prod_{i=1}^{\infty} (1 + t^i)^{\kappa_i} \text{ where } \kappa_i = 0, 1 \text{ or } 2, \text{ and } \kappa_i \text{ variate in some cycle.}$$

Moreover, we construct the generalized version of Bessenrodt's algorithms which refine the extended Andrews-Olsson identity. From these algorithms, we can give crystal structures on (pair of) the (sub)set(s) of strict partitions which are isomorphic to  $B(\Lambda)$ .

## INTRODUCTION

A weakly decreasing sequence of nonnegative integers  $\lambda = (\lambda_1, \lambda_2, \dots)$  is called a *partition* of  $m$ , denoted by  $\lambda \vdash m$ , if  $m = \sum_i \lambda_i$ . A partition  $\lambda$  is called a *strict partition* if all parts are strictly decreasing and an *odd partition* if all parts are odd. Let  $\mathcal{P}[m]$  (respectively,  $\mathcal{S}[m]$  and  $\mathcal{O}[m]$ ) be the set of (respectively, strict and odd) partitions of  $m$ . Denote by  $\mathcal{P}$  (respectively,  $\mathcal{S}$  and  $\mathcal{O}$ ) the set of all (respectively, strict and odd) partitions.

**Theorem 0.1** (Euler's partition theorem). *For all  $m \in \mathbb{Z}_{\geq 0}$ ,*

$$|\mathcal{S}[m]| = |\mathcal{O}[m]|.$$

He proved the identity by showing that their generating functions coincide with each other as follows:

$$\prod_{i=1}^{\infty} (1 + t^i) = \prod_{i=1}^{\infty} \frac{1}{(1 - t^{2i-1})}.$$

In 1882, Sylvester showed the identity in an alternative way. He proved the identity by constructing combinatorial algorithm which gives an bijection between  $\mathcal{O}[m]$  and  $\mathcal{S}[m]$ .

The generalization of Euler's partition theorem is still one of the main topics in combinatorial theory [3, 17, 18, 23, 24]. Thus, for subsets  $\mathcal{A}$  and  $\mathcal{B}$  of partitions,

showing  $|\mathcal{A}[m]| = |\mathcal{B}[m]|$  and constructing a combinatorial bijection between  $\mathcal{A}[m]$  and  $\mathcal{B}[m]$ ,

are interesting problems. In 1991 [1], Andrews and Olsson defined the series of subsets of partitions,  $\mathcal{AO}_1^{X_N}$  and  $\mathcal{AO}_2^{X_N}$  ( $N \in \mathbb{Z}_{>0}$ ), and showed that

$$|\mathcal{AO}_1^{X_N}[m]| = |\mathcal{AO}_2^{X_N}[m]| \text{ for all } m \in \mathbb{Z}_{\geq 0}.$$

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Shortly after, Bessenrodt constructed a combinatorial insertion algorithm which gives a bijection between them. Until recently, the Andrews-Olsson identity is generalized and developed [2, 4, 26]. On the other hand, the notion of partitions has been generalized such as overpartitions, multi-colored partitions. Using these notions, many mathematicians interpreted or proved combinatorial identities arising from hypergeometric series [5, 6, 25].

The characters of integrable modules over quantum groups  $U_q(\mathfrak{g})$  are important algebraic invariant which *determines* the isomorphism classes in the sense that  $M \cong N$  if and only if  $\text{ch}M \cong \text{ch}N$ . In [14, 15], Kashiwara developed the *crystal basis theory* for integrable  $U_q(\mathfrak{g})$ -modules from which a lot of combinatorial properties of integrable module can be deduced. By using realization of crystal bases, one can compute the characters of integrable modules (see Section 1).

In [7], Hayashi gave a  $U_q(A_n^{(1)})$ -module structure on the space of *Young diagrams*, which can be understood as the set of all partitions with coloring. In [21], Misra and Miwa showed that the *reduced* Young diagrams can be realized as a crystal basis of basic representation and has a crystal structure induced by the Kashiwara operators  $\tilde{e}_i, \tilde{f}_i$ . In [11], Kang introduced the notion of *Young walls* (which can be understood as a generalization of Young diagrams) as a new combinatorial scheme for realizing the crystal bases of integrable highest weight modules over all classical quantum affine algebras. In that paper, it was shown that the set  $Z(\Lambda)$  of *proper* Young walls has the crystal structure. Moreover, he proved that the crystal  $B(\Lambda)$  of the basic representation  $V(\Lambda)$  is realized as the crystal  $Y(\Lambda)$  consisting of *reduced* Young walls. In [12, 13], Kang and Kwon gave the  $U_q(\mathfrak{g})$ -module structure on the space of proper Young walls and showed the decomposition formulae for the space into highest weight modules. Using these realizations, we can derive the explicit formulae for the characters of level 1 highest weight modules (see [8, 12] for more details).

In this paper, we extend the Andrews-Olsson identity and Bessenrodt's insertion algorithm to the *two-colored partition* setting by regarding Young walls as two-colored partitions. We denote by the sets of two-colored partitions  $\mathcal{AO}_1(\Lambda)$  corresponding to the sets of reduced Young walls and define other sets of two-colored partitions  $\mathcal{AO}_2(\Lambda)$ , which can be also realized as (pair of) the (sub)set(s) of strict partitions. Then we first show that the number of two-colored partitions of  $m \in \mathbb{Z}_{\geq 0}$  in  $\mathcal{AO}_1(\Lambda)$  and  $\mathcal{AO}_2(\Lambda)$  coincides with each other by developing new combinatorial algorithm (see Section 3). As a corollary, we can compute the generating functions of those sets and hence of the sets of reduced Young walls as follows:

Type	$\Lambda$	generating function
$A_{2n}^{(2)}$	$\Lambda_0$	$\prod_{i=1}^{\infty} (1 + t^i)^{\kappa_i}, \quad \kappa_i = 0 \text{ if } i \equiv 0 \pmod{2n+1} \text{ and } \kappa_i = 1, \text{ otherwise.}$
$A_{2n-1}^{(2)}$	$\Lambda_0, \Lambda_1$	$\prod_{i=1}^{\infty} (1 + t^i)$
$B_n^{(1)}$	$\Lambda_0, \Lambda_1$	$\prod_{i=1}^{\infty} (1 + t^i)^{\kappa_i}, \quad \kappa_i = 2 \text{ if } i \equiv 0 \pmod{2n} \text{ and } \kappa_i = 1, \text{ otherwise.}$
	$\Lambda_n$	$\prod_{i=1}^{\infty} (1 + t^i)^{\kappa_i}, \quad \kappa_i = 2 \text{ if } i \equiv n \pmod{2n} \text{ and } \kappa_i = 1, \text{ otherwise.}$
$D_n^{(1)}$	$\Lambda_0, \Lambda_1, \Lambda_{n-1}, \Lambda_n$	$\prod_{i=1}^{\infty} (1 + t^i)^{\kappa_i}, \quad \kappa_i = 2 \text{ if } i \equiv 0 \pmod{n-1} \text{ and } \kappa_i = 1, \text{ otherwise.}$
$D_{n+1}^{(2)}$	$\Lambda_0, \Lambda_n$	$\prod_{i=1}^{\infty} (1 + t^i)$

Note that those formulae can be interpreted by the *principally characters* and were studied in [10] by using an algebraic method. Using a vertex operator technique, Nakajima and Yamada also proved the identities for types of  $D_{n+1}^{(2)}$  and  $A_{2n}^{(2)}$  [22]. The identity between  $\mathcal{AO}_1(\Lambda)$  and  $\mathcal{AO}_2(\Lambda)$  is extended to more general form (see Corollary 3.9).

After that, we construct bijection between  $\mathcal{AO}_1(\Lambda)$  and  $\mathcal{AO}_2(\Lambda)$  by applying and generalizing the Bessenrodt's insertion algorithm (see Section 4). To do this, we use the notion of overpartition to handle the set  $\mathcal{AO}_1(\Lambda)$ . More precisely, we *twist* the set  $\mathcal{AO}_1(\Lambda)$  by replacing some subsequences of  $\lambda \in \mathcal{AO}_1(\Lambda)$  to associated overpartitions. From these bijections, we can give crystal structures on the alternative combinatorial

objects which realize the crystals  $B(\Lambda)$  (see Appendix B for  $D_3^{(2)}$ ). For the forthcoming works, we would like to study the crystal structure on the alternative sets without the bijections.

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## 1. THE QUANTUM AFFINE ALGEBRAS

Let  $I = \{0, 1, \dots, n\}$  be the index set. The *affine Cartan datum*  $(A, P^\vee, P, \Pi^\vee, \Pi)$  consists of

- (1)  $A = (a_{ij})_{i,j \in I}$ , the *generalized Cartan matrix* of affine type,
- (2) a free abelian group  $P^\vee = \bigoplus_{i=0}^n \mathbb{Z}h_i \oplus \mathbb{Z}d$ , the *dual weight lattice*,
- (3) a free abelian group  $P = \bigoplus_{i=0}^n \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta \subset \mathfrak{h}^* = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$ , the *weight lattice*,
- (4)  $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee := \text{Hom}(P, \mathbb{Z})$ , the set of *simple coroots*,
- (5)  $\Pi = \{\alpha_i \in P \mid i \in I\}$ , the set of *simple roots*.

We denote by  $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geq 0}, i \in I\}$  the set of *dominant integral weights*. The free abelian group  $Q := \sum_{i \in I} \mathbb{Z}\alpha_i$  is called the *root lattice* and we denote by  $Q^+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ . For  $\alpha = \sum_{i \in I} k_i \alpha_i \in Q^+$ , we define the *height* of  $\alpha$  to be  $\text{ht}(\alpha) := \sum_{i \in I} k_i$ . Note that the affine Cartan matrix is *symmetrizable*; i.e., there is a diagonal matrix  $D = \text{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$  such that  $DA$  is symmetric.

Let  $q$  be an indeterminate. For  $i \in I$  and  $m, n \in \mathbb{Z}_{\geq 0}$ , define

$$q_i = q^{s_i}, \quad [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_{q_i}! = \prod_{k=1}^n [k]_{q_i}, \quad \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} = \frac{[m]_{q_i}!}{[m-n]_{q_i}! [n]_{q_i}!}.$$

**Definition 1.1.** The *quantum group*  $U_q(\mathfrak{g}_n)$  with an affine Cartan datum  $(A, P^\vee, P, \Pi^\vee, \Pi)$  is the associative algebra over  $\mathbb{Q}(q)$  with **1** generated by  $e_i, f_i$  ( $i \in I$ ) and  $q^h$  ( $h \in P^\vee$ ) satisfying the following relations:

- (1)  $q^0 = 1, q^h q^{h'} = q^{h+h'}$  for  $h, h' \in P^\vee$ ,
- (2)  $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$  for  $h \in P^\vee, i \in I$ ,
- (3)  $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ , where  $K_i = q_i^{h_i}$ ,
- (4)  $\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = \sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$  if  $i \neq j$ .

A  $U_q(\mathfrak{g}_n)$ -module  $V$  is called a *weight module* if it admits a *weight space decomposition*  $V = \bigoplus_{\mu \in P} V_\mu$ , where  $V_\mu = \{v \in V \mid q^h v = q^{\langle h, \mu \rangle} v \text{ for all } h \in P^\vee\}$ . If  $\dim_{\mathbb{Q}(q)} V_\mu < \infty$  for all  $\mu \in P$ , we define the *character* of  $V$  by

$$\chi_{\mathfrak{g}_n}(V) = \sum_{\mu \in P} (\dim_{\mathbb{Q}(q)} V_\mu) e(\mu).$$

Then it is proved in [19], [8, Chapter 3] that the category of integrable modules is semisimple. Here, the irreducible objects are isomorphic to certain modules  $V(\Lambda)$  which are labeled by  $\Lambda \in P^+$ . For  $V(\Lambda)$  ( $\Lambda \in P^+$ ), we set  $\chi_{\mathfrak{g}_n}^\Lambda = \chi_{\mathfrak{g}_n}(V(\Lambda))$ .

**Definition 1.2.** We define the *principally specialized character* of  $V(\Lambda)$  as follows:

$$\chi_{\mathfrak{g}_n}^\Lambda(t) = \chi_{\mathfrak{g}_n}^\Lambda|_{\substack{e(\Lambda)=1 \\ e(-\alpha_i)=t}}, \text{ for all } i \in I.$$

where  $t$  is the indeterminate.

The *level* of  $\Lambda \in P^+$  is defined to be the nonnegative integer  $\langle c, \Lambda \rangle$ , where  $c$  is the *center* of  $\mathfrak{g}_n$ . Let  $\delta = d_0\alpha_0 + \cdots + d_n\alpha_n$  be the *null root* of  $\mathfrak{g}_n$  and set  $a_i = d_i$  if  $\mathfrak{g}_n \neq D_{n+1}^{(2)}$ ,  $a_i = 2d_i$  if  $\mathfrak{g}_n = D_{n+1}^{(2)}$ . Set  $\Delta = \sum_{i \in I} a_i$ . Then  $\Delta$  becomes an integer depending on  $\mathfrak{g}_n$  as follows:

$$(1.1) \quad \begin{array}{|c|c|c|c|c|c|} \hline \text{Type} & A_{2n}^{(2)} & A_{2n-1}^{(2)} & B_n^{(1)} & D_n^{(1)} & D_{n+1}^{(2)} \\ \hline \Delta & 2n+1 & 2n-1 & 2n & 2n-2 & 2n+2 \\ \hline \end{array}$$

Let  $\mathbb{A}_0 = \{f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0\}$ . It is shown in [15] that  $V(\Lambda)$  has a unique *crystal basis*  $(L(\Lambda), B(\Lambda))$ , where  $L(\Lambda)$  is a free  $\mathbb{A}_0$ -lattice of  $V(\Lambda)$ ,  $B(\Lambda)$  is a  $\mathbb{Q}$ -basis of  $L(\Lambda)/qL(\Lambda)$ .  $B(\Lambda)$  has an  $I$ -colored oriented graph structure induced by the *Kashiwara operators*  $\tilde{e}_i, \tilde{f}_i$  ( $i \in I$ ). Moreover,  $B(\Lambda)$  encodes the combinatorial information of  $V(\Lambda)$  as follows:

$$\bullet B(\Lambda) = \bigsqcup_{\mu \in P} B(\Lambda)_\mu \text{ where } B(\Lambda)_\mu = B(\Lambda) \cap V(\Lambda)_\mu, \quad \bullet |B(\Lambda)_\mu| = \dim_{\mathbb{Q}(q)} V(\Lambda)_\mu.$$

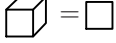

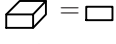

Thus  $\chi_{\mathfrak{g}_n}^\Lambda$  and  $\chi_{\mathfrak{g}_n}^\Lambda(t)$  can be expressed as follows:

$$(1.2) \quad \chi_{\mathfrak{g}_n}^\Lambda = \sum_{\mu \in P} |B(\Lambda)_\mu| e(\mu), \quad \chi_{\mathfrak{g}_n}^\Lambda(t) = \sum_{m \in \mathbb{Z}_{\geq 0}} \left( \sum_{\substack{\mu \in P \\ \text{ht}(\Lambda - \mu) = m}} |B(\Lambda)_\mu| \right) t^m.$$

In particular,  $B(\Lambda)$  becomes a  $U_q(\mathfrak{g}_n)$ -crystal. (See [8, 16] for more details on the crystal bases theory.)

## 2. YOUNG WALLS AND VARIOUS PARTITIONS

**2.1. Young walls.** In [11], Kang gave realizations of level 1 highest weight crystals  $B(\Lambda)$  for all classical quantum affine algebras in terms of *reduced Young walls* (see [9] as well). From now on, we assume that  $\mathfrak{g}_n$  is of type  $A_{2n}^{(2)}, A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}$  or  $D_{n+1}^{(2)}$ . Basically, the Young walls are built of colored blocks. There are three types of blocks whose shapes are different and which appear depending on type as follows:

Shape	Width	Thickness	Height	Type
 = 	1	1	1	all types
 = 	1	1	1/2	$A_{2n}^{(2)}, B_n^{(1)}, D_{n+1}^{(2)}$
 =  ,  = 	1	1/2	1	$A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}$

Given  $\mathfrak{g}_n$  and a dominant integral weight  $\Lambda$  of level 1, we fix a frame  $Y_\Lambda$  called the *ground-state Young wall* of weight  $\Lambda$ . The set of Young walls is the set of blocks built on the ground-state Young wall by the following rules:

- All blocks should be placed on the top of the ground-state Young wall or another block.
- The colored blocks should be stacked in the given pattern depending  $\mathfrak{g}_n$  and  $\Lambda$ .
- No block can be placed on the top of a column of half-thickness.
- Except for the right-most column, there should be no free space to the right of any blocks.

We will give the patterns in Appendix A. We write young wall  $Y = (y_k)_{k=1}^\infty$  as an infinite sequence of its columns where the columns are enumerated from right to left. A column in Young wall is called a *full column* if its height is a multiple of unit length and its top is of unit thickness. We say a Young wall *proper* if none of the full columns have same height. The part of a column consisting of  $a_i$ -many  $i$ -blocks, for each  $i \in I$ , in some cyclic order is called a  $\delta$ -column.

### Definition 2.1.

- A column in a proper Young wall is said to contain a *removable*  $\delta$  if we may remove a  $\delta$ -column from  $Y$  and still obtain a proper Young wall.

- (2) A proper Young wall is said to be *reduced* if none of its columns contain a removable  $\delta$ .

For a given Young wall  $Y$ , we define the *weight*,  $\text{wt}(Y)$ , of  $Y$  as follows:

$$(2.1) \quad \text{wt}(Y) = \Lambda - \sum_{i \in I} m_i \alpha_i,$$

where  $m_i$  is the number of  $i$ -blocks on the ground-state Young wall  $Y_\Lambda$ .

Let  $Z(\Lambda)$  be the set of all proper Young walls built on  $Y_\Lambda$  and  $Y(\Lambda)$  be the set of all reduced proper Young walls built on  $Y_\Lambda$ .

**Theorem 2.2.** [11]

- (1)  $Z(\Lambda)$  has a crystal structure induced by the Kashiwara operators  $\tilde{e}_i$  and  $\tilde{f}_i$ .
- (2) There is a crystal isomorphism between  $Y(\Lambda)$  and  $B(\Lambda)$ .

**Definition 2.3.** For an  $m \in \mathbb{Z}_{\geq 0}$ ,  $\mu \in P$  and subset  $A$  of  $Z(\Lambda)$ , we define

- (1)  $A[m]$  to be the subset of  $A$  which has  $m$  blocks on  $Y_\Lambda$ ,
- (2)  $A[\mu]$  to be the subset of  $A$  consisting of Young walls with weight  $\mu$ ,
- (3) the *virtual character*  $\overset{\circ}{\chi}_{\mathfrak{g}_n}(A)$  of  $A$  to be

$$\overset{\circ}{\chi}_{\mathfrak{g}_n}(A) = \sum_{\mu \in P} |A[\mu]| e(\mu).$$

Then the equations in (1.2) can be written by

$$(2.2) \quad \chi_{\mathfrak{g}_n}^\Lambda = \overset{\circ}{\chi}_{\mathfrak{g}_n}(Y(\Lambda)) = \sum_{\mu \in P} |Y(\Lambda)[\mu]| e(\mu), \quad \chi_{\mathfrak{g}_n}^\Lambda(t) = \sum_m |Y(\Lambda)[m]| t^m.$$

Set  $\epsilon = 2$  if  $\mathfrak{g}_n$  is type of  $D_{n+1}^{(2)}$ , and  $\epsilon = 1$ , otherwise.

**Proposition 2.4.** [12, Corollary 2.5]

$$Z(\Lambda) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} B(\Lambda - \epsilon k \delta)^{\oplus |\mathcal{P}(k)|},$$

where  $B(\Lambda - m\delta)$  is the crystal of the highest weight module  $V(\Lambda - m\delta)$ .

In terms of Young walls, Proposition 2.4 can be interpreted as follows:

$$(2.3) \quad |Z(\Lambda)[m]| = \sum_{k \geq 0, m - k\Delta \geq 0} (|Y(\Lambda)[m - k\Delta]| \times |\mathcal{P}(k)|).$$

**2.2. Various partitions.** For  $\mathbb{N} := \mathbb{Z}_{>0}$ , let  $\overline{\mathbb{N}}$  be the set of positive integers which are overlined. Set  $N = \mathbb{N} \sqcup \overline{\mathbb{N}} \sqcup \{0\}$ . We give a linear order  $\succ$  on  $N$  by defining  $n \succ \overline{n}$  for  $n \in \mathbb{N}$  and define a map  $c : N \setminus \{0\} \rightarrow \mathbb{Z}_2$  by  $c(\overline{n}) = 1$  and  $c(n) = 0$  for  $n \in \mathbb{N}$ .

With the linear order  $\succ$  on  $N$ , we can naturally define the notion of the set of partitions  $\mathcal{P}^2$  consisting of sequences in  $N$  and  $\lambda \in \mathcal{P}^2$  is called a *two-colored partition*. For a two-colored partition  $\lambda = (k^{m_k}, \overline{k}^{m_{\overline{k}}} \dots, 2^{m_2}, \overline{2}^{m_{\overline{2}}}, 1^{m_1}, \overline{1}^{m_{\overline{1}}}) \in \mathcal{P}^2$ , we say that  $\lambda$  is a two-colored partition of  $m \in \mathbb{Z}_{\geq 0}$ , if  $\sum_{i=1}^k i(m_i + m_{\overline{i}}) = m$ , and write  $\Sigma_\lambda = m$ . For each subset  $\mathcal{A}$  of  $\mathcal{P}^2$  and  $m \in \mathbb{Z}_{\geq 0}$ , we denote by  $\mathcal{A}[m] := \{\lambda \in \mathcal{A} \mid \Sigma_\lambda = m\}$ .

For  $a \succ b \in N$  such that  $\overline{a} \neq b$ , we define  $a - b$  to be the element in  $N$  whose value is given by the ordinary subtraction of their values, and  $c(a - b) \equiv c(a) - c(b) \pmod{2}$ . In particular, if  $\overline{a} = b$ , we define  $a - b = 0$ . Similarly, we can define  $a + b$ , for  $a, b \in N$ . For example,  $\overline{3} - 2 = \overline{1}$ ,  $\overline{3} + 2 = \overline{5}$ ,  $\overline{2} - 2 = 0$  and

$\bar{2} + 0 = \bar{2}$ . We also define  $\mathbb{Z}_{\geq 0}$ -multiplication  $\cdot : \mathbb{Z}_{\geq 0} \times N \rightarrow N$  by  $(k, n) \mapsto k \cdot n$  with  $c(n) = c(k \cdot n)$ . For instance,  $2 \cdot \bar{2} = \bar{4}$ .

For sequences  $\lambda = (\lambda_1, \dots, \lambda_t)$  and  $\mu = (\mu_1, \dots, \mu_l)$  of  $N$ , we define

- (i)  $\ell(\lambda) = t$ , the *length* of  $\lambda$ ,
- (ii)  $\mu * \lambda = (\mu_1, \dots, \mu_l, \lambda_1, \dots, \lambda_t)$ , the *concatenation* of  $\mu$  and  $\lambda$
- (iii)  ${}^\circ \lambda = (\lambda_t, \lambda_{t-1}, \dots, \lambda_1)$  the *reverse* of  $\lambda$ .

**Definition 2.5.** [5]

- (1) An *overpartition* is a non-increasing sequence of  $\mathbb{N}$  in which the first occurrence of the number may be overlined. Let  $\mathcal{P}^o$  be the set of all overpartitions.
- (2) Let  $\mathcal{P}^t$  be the set of elements in  $\mathcal{P}^2$  satisfying that, for each  $m \in \mathbb{N}$ , both  $m$  and  $\bar{m}$  do not appear as a part, simultaneously.

For instance, the number of overpartitions of 3 is 8:

$$(3), (\bar{3}), (2, 1), (\bar{2}, 1), (2, \bar{1}), (\bar{2}, \bar{1}), (1, 1, 1), (\bar{1}, 1, 1).$$

**Definition 2.6.** For a given proper Young wall  $Y = (y_k)_{k=1}^\infty$ , we define several sequences associated to  $Y$ ,

- (1)  $P(Y) = (p(y_k))_{k=1}^\infty$  to be the sequence in  $\mathbb{Z}_{\geq 0}$ , where the  $p(y_k)$  is the number of blocks in  $k$ -th column of  $Y$  except the one in  $Y_\Lambda$ ,
- (2)  $P^t(Y) = (p^t(y_k))_{k=1}^\infty$  to be the sequence in  $N$ , where the  $p^t(y_k)$  is  $\overline{p(y_k)}$ , if the top of the  $k$ -th column is half-thickness block and placed in the front side, and  $p(y_k)$  otherwise.

For  $m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}$ , we define  $N\{(m_1, m_2)\} := \mathbb{N} \cup (m_1 \cdot \bar{\mathbb{N}} + m_2) \cup \{\bar{m}_1, 0\}$  and  $\mathcal{S}^t\{(m_1, m_2); m_3\}$  to be the subset of  $\mathcal{P}^t$  whose parts are elements in  $N\{(m_1, m_2)\}$  and only  $k \cdot m_1 + m_2$ ,  $k \cdot \bar{m}_1 + m_2$ ,  $m_1$ ,  $\bar{m}_1$  and  $k \cdot m_3$  ( $k \in \mathbb{Z}_{\geq 0}$ ) can be repeated.

For each type of  $\mathfrak{g}_n$  and  $\Lambda$ , we define  $W^1 = (w_1^1, w_2^1, w_3^1)$  and  $W^2 = (w_1^2, w_2^2)$ :

Type	$\Lambda$	$W^1$	$W^2$
$A_{2n}^{(2)}$	$\Lambda_0$	$(0, 0, 2n+1)$	$(2n+1, 2n+1)$
$A_{2n-1}^{(2)}$	$\Lambda_0, \Lambda_1$	$(2n-1, 0, 2n-1)$	$(2n-1, 2n-1)$
$B_n^{(1)}$	$\Lambda_0, \Lambda_1$	$(2n, 0, n)$	$(2n, 2n)$
$B_n^{(1)}$	$\Lambda_n$	$(n, 2n, n)$	$(2n, 2n)$
$D_n^{(1)}$	$\Lambda_0, \Lambda_1, \Lambda_{n-1}, \Lambda_n$	$(n-1, 0, n-1)$	$(2n-2, n-1)$
$D_{n+1}^{(2)}$	$\Lambda_0, \Lambda_n$	$(0, 0, n+1)$	$(2n+2, n+1)$

For each given  $W^1$  and  $W^2$ , we define the subsets  $\mathcal{Z}(\Lambda)$  and  $\mathcal{AO}_i(\Lambda)$  ( $i = 1, 2$ ) of  $\mathcal{P}^t$  as follows:

- (1)  $\mathcal{Z}(\Lambda) := \mathcal{S}^t\{(w_1^1, w_2^1); w_3^1\}$ ,
- (2) Let  $\mathcal{AO}_1(\Lambda)$  to be the subset of  $\mathcal{Z}(\Lambda)$  satisfying
  - the difference between successive parts is at most  $w_1^2$  and the smallest parts is less than  $w_1^2$ ,
  - the difference between successive parts is strictly less than  $w_1^2$  if either parts is congruent to  $w_3^1, \bar{w}_3^1$  or 0 modulo  $w_2^2$ .
- (3) Let  $\mathcal{AO}_2(\Lambda)$  to be the subset of partitions in  $\mathcal{Z}(\Lambda)$  such that

Type	$\mathcal{AO}_2(\Lambda)$
$A_{2n-1}^{(2)}$	each part is not a multiple of $w_1^2$ and no part can be repeated.
$A_{2n}^{(2)}, B_n^{(1)}, D_n^{(1)}$	each part is not a multiple of $w_1^2$ .
$D_{n+1}^{(2)}$	no part can be repeated.

**Remark 2.7.**

- (1) One can check that the map  $P^t : Z(\Lambda) \rightarrow \mathcal{P}^2$  is injective and the image coincides with  $\mathcal{Z}(\Lambda)$ . Moreover, the image of  $Y(\Lambda)$  under this map coincides with  $\mathcal{AO}_1(\Lambda)$ , also.
- (2) The map  $P : Z(\Lambda) \rightarrow \mathcal{P}$  is injective for the types  $D_{n+1}^{(2)}$  and  $A_{2n}^{(2)}$ . But in general,  $P : Z(\Lambda) \rightarrow \mathcal{P}$  is not injective.
- (3) For the types  $D_{n+1}^{(2)}$ ,  $A_{2n-1}^{(2)}$  and  $A_{2n}^{(2)}$ , we can identify the sets  $\mathcal{AO}_2(\Lambda)$  with the set of strict partitions  $\mathcal{S}$  if  $\mathfrak{g}_n$  is  $D_{n+1}^{(2)}$  or  $A_{2n-1}^{(2)}$ , and the subset of strict partitions  $\mathcal{S}_{(w_1^2)}$  whose elements do not contain multiples of  $w_1^2$  as a part if  $\mathfrak{g}_n$  is  $A_{2n}^{(2)}$ .

For the rest of this section, we assume that  $\mathfrak{g}_n$  is of type  $A_{2n-1}^{(2)}$ ,  $B_n^{(1)}$  ( $\Lambda \neq \Lambda_n$ ) and  $D_n^{(1)}$ .

Let  $\lambda$  to be elements in  $\mathcal{AO}_1(\Lambda)$ . For a subsequence  $\underline{\lambda} = (\lambda_i, \dots, \lambda_{i+t})$ , we say  $\underline{\lambda}$  is a  $w_1^2$ -subsequence of  $\lambda$  if it satisfies

- for all  $i \leq j \leq i+t$ ,  $\lambda_i = k \cdot w_1^2$  or  $k \cdot \overline{w_1^2}$ ,
- for all  $i \leq j < i+t$ ,  $\lambda_i - \lambda_{i-1} = 0$  or  $\overline{w_1^2}$ .

For  $k \in \mathbb{Z}_{>0}$ , we say  $\underline{\lambda}$  is the  $k \cdot w_1^2$ -component of  $\lambda$  if it is the maximal one among  $w_1^2$ -subsequence of  $\lambda$  containing  $k \cdot w_1^2$  or  $k \cdot \overline{w_1^2}$  as a part.

Let  $Y \in Y(\Lambda)$ . For  $\lambda = P^t(Y)$ ,  $k \cdot w_1^2$ -component  $\underline{\lambda}$  of  $\lambda$  can be written by one of the followings:

- (i)  $\underline{\lambda} = ((k_l \cdot \overline{w_1^2})^{t_l}, (k_{l-1} \cdot w_1^2)^{t_{l-1}}, \dots, (k_2 \cdot \overline{w_1^2})^{t_2}, (k_1 \cdot \overline{w_1^2})^{t_1})$ ,
- (ii)  $\underline{\lambda} = ((k_l \cdot \overline{w_1^2})^{t_l}, (k_{l-1} \cdot w_1^2)^{t_{l-1}}, \dots, (k_2 \cdot \overline{w_1^2})^{t_2}, (k_1 \cdot w_1^2)^{t_1})$ ,
- (iii)  $\underline{\lambda} = ((k_l \cdot w_1^2)^{t_l}, (k_{l-1} \cdot \overline{w_1^2})^{t_{l-1}}, \dots, (k_2 \cdot w_1^2)^{t_2}, (k_1 \cdot \overline{w_1^2})^{t_1})$ ,
- (iv)  $\underline{\lambda} = ((k_l \cdot w_1^2)^{t_l}, (k_{l-1} \cdot \overline{w_1^2})^{t_{l-1}}, \dots, (k_2 \cdot w_1^2)^{t_2}, (k_1 \cdot w_1^2)^{t_1})$ ,

where  $k_l > 0$  and  $k_{i+1} = k_i - 1$ . Let  $\tilde{\lambda}$  be the sequence in  $\mathcal{P}^o$  associated with  $\underline{\lambda}$  by

- (i)  $\tilde{\lambda} = (k_l \cdot \overline{w_1^2}, (k_l \cdot w_1^2)^{t_l-1}, k_{l-1} \cdot \overline{w_1^2}, (k_{l-1} \cdot w_1^2)^{t_{l-1}-1}, \dots, k_2 \cdot \overline{w_1^2}, (k_2 \cdot w_1^2)^{t_2-1}, k_1 \cdot \overline{w_1^2}, (k_1 \cdot w_1^2)^{t_1-1})$ ,
- (ii)  $\tilde{\lambda} = (k_l \cdot \overline{w_1^2}, (k_l \cdot w_1^2)^{t_l-1}, k_{l-1} \cdot \overline{w_1^2}, (k_{l-1} \cdot w_1^2)^{t_{l-1}-1}, \dots, k_2 \cdot \overline{w_1^2}, (k_2 \cdot w_1^2)^{t_2-1}, k_1 \cdot \overline{w_1^2}, (k_1 \cdot w_1^2)^{t_1-1})$ ,
- (iii)  $\tilde{\lambda} = ((k_l \cdot w_1^2)^{t_l}, k_{l-1} \cdot \overline{w_1^2}, (k_{l-1} \cdot w_1^2)^{t_{l-1}-1}, \dots, k_2 \cdot \overline{w_1^2}, (k_2 \cdot w_1^2)^{t_2-1}, k_1 \cdot \overline{w_1^2}, (k_1 \cdot w_1^2)^{t_1})$ ,
- (iv)  $\tilde{\lambda} = ((k_l \cdot w_1^2)^{t_l}, k_{l-1} \cdot \overline{w_1^2}, (k_{l-1} \cdot w_1^2)^{t_{l-1}-1}, \dots, k_2 \cdot \overline{w_1^2}, (k_2 \cdot w_1^2)^{t_2-1}, k_1 \cdot \overline{w_1^2}, (k_1 \cdot w_1^2)^{t_1})$ .

For a  $\lambda \in \mathcal{AO}_1(\Lambda)$ , we define  $P^o(\lambda)$  to be the element of  $\mathcal{P}^2$  by replacing all  $k \cdot w_1^2$ -components  $\underline{\lambda}$  of  $\lambda$  to their associated sequences  $\tilde{\lambda}$  in  $\mathcal{P}^o$ . Then we have an injective map  $P^o : \mathcal{AO}_1(\Lambda) \rightarrow \mathcal{P}^2$  and can recover  $\lambda$  out of  $P^o(\lambda)$ .

**Definition 2.8.** We denote by  $\mathcal{AO}_{1^o}(\Lambda)$  the set of all  $P^o(\lambda)$  for  $\lambda \in \mathcal{AO}_1(\Lambda)$ .

For  $\lambda \in \mathcal{AO}_2(\Lambda)$ ,  $k \cdot w_1^2$ -component  $\check{\lambda}$  of  $\lambda$  can be written by one of the followings:

- (a)  $\check{\lambda} = ((k_{2n+2} \cdot \overline{w_1^2})^{t_{2n+2}}, (k_{2n+1} \cdot \overline{w_1^2})^{t_{2n+1}}, \dots, (k_2 \cdot \overline{w_1^2})^{t_2}, (k_1 \cdot \overline{w_1^2})^{t_1})$ ,
- (b)  $\check{\lambda} = ((k_{2n+1} \cdot \overline{w_1^2})^{t_{2n+1}}, (k_{2n} \cdot \overline{w_1^2})^{t_{2n}}, \dots, (k_2 \cdot \overline{w_1^2})^{t_2}, (k_1 \cdot \overline{w_1^2})^{t_1})$ ,

for some  $n \in \mathbb{Z}_{\geq 0}$ . Let  $\ddot{\lambda}$  be the sequence in  $\mathcal{P}^2$  associated the  $k \cdot w_1^2$ -component  $\check{\lambda}$  which is defined as follows:

- (a)  $\ddot{\lambda} = ((k_{2n+2} \cdot w_1^2)^{t_{2n+2}}, (k_{2n+1} \cdot \overline{w_1^2})^{t_{2n+1}}, \dots, (k_2 \cdot w_1^2)^{t_2}, (k_1 \cdot \overline{w_1^2})^{t_1})$ ,
- (b)  $\ddot{\lambda} = ((k_{2n+1} \cdot \overline{w_1^2})^{t_{2n+1}}, (k_{2n} \cdot w_1^2)^{t_{2n}}, \dots, (k_2 \cdot w_1^2)^{t_2}, (k_1 \cdot \overline{w_1^2})^{t_1})$ ,

For a given partition  $\lambda \in \mathcal{AO}_2(\Lambda)$ , we define  $P^s(\lambda)$  to be the element of  $\mathcal{P}^2$  by replacing all  $k \cdot w_1^2$ -components  $\check{\lambda}$  of  $\lambda$  to their associated sequences  $\ddot{\lambda}$  in  $\mathcal{P}^o$ . Then we have an injective map  $P^s : \mathcal{AO}_2(\Lambda) \rightarrow \mathcal{P}^2$  and can recover  $\lambda$  out of  $P^s(\lambda)$ .

**Definition 2.9.** We denote by  $\mathcal{AO}_{2^s}(\Lambda)$  the set of all  $P^s(\lambda)$  for  $\lambda \in \mathcal{AO}_2(\Lambda)$ .



## 3. THE ANDREWS-OLSSON IDENTITY

In this section, we prove the following theorem. *Hereafter, we drop  $(\Lambda)$  for notations given in Section 2.*

**Theorem 3.1.** *For all  $m \in \mathbb{Z}_{>0}$ , we have*

$$|\mathcal{AO}_1[m]| = |\mathcal{AO}_2[m]|.$$

For  $i = 1, 2$ , we denote by  $\mathcal{AO}_i^c$  the complement of  $\mathcal{AO}_i$  in  $\mathcal{Z}$ .

**Proposition 3.2.** *For all  $m \in \mathbb{Z}_{>0}$ , there are 1 – 1 and onto maps given as follows:*

$$\overline{\Psi}[m] : \mathcal{AO}_1^c[m] \rightarrow \bigsqcup_{k>0, m-kw_1^2 \geq 0} (\mathcal{AO}_1[m - kw_1^2] \times \mathcal{P}[k]).$$

*Proof.* In this proof, we will give an explicit 1 – 1 and onto map between them. Let  $\Psi[m]$  be a map from  $\mathcal{AO}_1^c[m]$  to  $\bigsqcup_{k>0, m-kw_1^2 \geq 0} \mathcal{AO}_1[m - kw_1^2]$  by the following algorithm **(A)**:

**(A1)** Let  $Y = (y_1, y_2, \dots) \in \mathcal{AO}_1^c[m]$  be given. Set  $Y^{(0)} = Y$  and  $l = 0$ .

**(A2)** Find a maximal  $i$  such that

$$(3.1) \quad y_{i-1}^{(l)} - y_i^{(l)} \succeq t \cdot w_1^2 \text{ for some } t \in \mathbb{Z}_{>0} \text{ and } (y_{i-1}^{(l)} - t \cdot w_1^2, y_i^{(l)}) \in \mathcal{Z}.$$

**(A3)** Among  $t$ 's satisfying the condition in (3.1), choose a maximal one and say  $t_{l+1}$ . Set

$$Y^{(l+1)} := (y_1^{(l)} - t_{l+1} \cdot w_1^2, y_2^{(l)} - t_{l+1} \cdot w_1^2, \dots, y_{i-1}^{(l)} - t_{l+1} \cdot w_1^2, y_i^{(l)}, y_{i+1}^{(l)}, \dots).$$

**(A4)** If  $Y^{(l+1)} \in \mathcal{AO}_1$ , then define  $Y' = Y^{(l+1)}$  and terminate this algorithm, otherwise  $l = l + 1$  and go to **(A2)**.

Then this algorithm terminates in a finite step and the followings hold:

$$k := \frac{\Sigma_Y - \Sigma_{Y'}}{w_1^2} \in \mathbb{Z}_{>0}, \quad Y' \in \mathcal{AO}_1[m - kw_1^2], \quad \lambda_i := \frac{y_i - y'_i}{w_1^2} \in \mathbb{Z}_{\geq 0}, \quad \lambda := (\lambda_1, \lambda_2, \dots) \in \mathcal{P}[k].$$

Thus we can get a function

$$\overline{\Psi}[m] : \mathcal{AO}_1^c[m] \rightarrow \bigsqcup_{k>0, m-kw_1^2 \geq 0} (\mathcal{AO}_1[m - kw_1^2] \times \mathcal{P}[k])$$

given by  $Y \mapsto (Y', \lambda)$ . Then  $\overline{\Psi}[m]$  becomes an 1 – 1 and onto map with the preimage of  $(Y', \lambda') \in \mathcal{AO}_1[m - kw_1^2] \times \mathcal{P}[k]$  given by  $Y$  as follows:

$$Y := (y'_1 + \lambda'_1 \cdot w_1^2, y'_2 + \lambda'_2 \cdot w_1^2, \dots, y'_k + \lambda'_k \cdot w_1^2, \dots).$$

□

**Definition 3.3.** For a given partition  $Y = (y_1, y_2, \dots) \in \mathcal{Z}(\Lambda)$  and  $e, j \in \mathbb{Z}_{>0}$ , we define the left inserting  $(\underbrace{e, \dots, e}_j)$  into  $Y$  to be the partition in  $\mathcal{Z}(\Lambda)$  denoted by  $(e)^j \hookrightarrow Y$  as follows:

$$(e)^j \hookrightarrow y := \begin{cases} (y_1, y_2, \dots, y_i, \underbrace{\bar{e}, \dots, \bar{e}}_j, y_{i+1}, \dots) & \text{if } y_i = \bar{e}, \\ (y_1, y_2, \dots, y_i, \underbrace{e, \dots, e}_j, y_{i+1}, \dots), & \text{otherwise,} \end{cases} \quad \text{for } y_i \succeq \bar{e} \succ y_{i+1}.$$

Recall that  $\epsilon = 2$ , if  $\mathfrak{g}_n$  is type of  $D_{n+1}^{(2)}$ , and  $\epsilon = 1$ , otherwise.



**Proposition 3.4.** *For all  $m \in \mathbb{Z}_{>0}$ , there are 1 – 1 and onto maps given as follows:*

$$\overline{\Phi}[m] : \mathcal{AO}_2^c[m] \rightarrow \bigsqcup_{k>0, m-kw_1^2 \geq 0} (\mathcal{AO}_2[m - kw_1^2] \times \mathcal{P}[k]).$$

*Proof.* Let  $\Phi[m]$  be a map from  $\mathcal{AO}_2^c[m]$  to  $\bigsqcup_{k>0, m-kw_1^2 \geq 0} \mathcal{AO}_2[m - kw_1^2]$  by the following algorithm (B):

(B1) Let  $Y \in \mathcal{AO}_2^c[m]$  be given. Set  $Y^{(0)} = Y$ ,  $\lambda^{(0)} = (0)$ ,  $\hat{\lambda}_0 = 0$  and  $l = 0$ .

(B2) Find a maximal  $i$  such that

$$y_i^{(l)} = \hat{\lambda}_{l+1} \cdot \epsilon^{-1}w_1^2 \text{ or } \hat{\lambda}_{l+1} \cdot \overline{\epsilon^{-1}w_1^2} \text{ for some } \hat{\lambda}_{l+1} \in \mathbb{Z}_{>0},$$

and set  $t_{l+1}$  to be the number of parts in  $Y^{(l)}$  which is equal to  $y_i^{(l)}$ .

(B3) Set  $a = \lfloor \epsilon^{-1}t_{l+1} \rfloor - c(y_i^{(l)})$  and define

$$Y^{(l+1)} := (y_1^{(l)}, y_2^{(l)}, \dots, y_{i-a}^{(l)}, y_{i+1}^{(l)}, \dots) \text{ and } \lambda^{(l+1)} = (\underbrace{\hat{\lambda}_{l+1}, \dots, \hat{\lambda}_{l+1}}_a) * \lambda^{(l)}.$$

(B4) If  $Y^{(l+1)} \in \mathcal{AO}_2$ , then define  $Y' = Y^{(l+1)}$  and terminate this algorithm, otherwise  $l = l + 1$  and go to (B2).

Then this algorithm terminates in a finite step and the followings hold:

$$k := \frac{\Sigma_Y - \Sigma_{Y'}}{w_1^2} \in \mathbb{Z}_{>0}, \quad Y' \in \mathcal{AO}_2[m - kw_1^2], \quad \lambda^{(l)} \in \mathcal{P}[k].$$

Thus we can get a function

$$\overline{\Phi}[m] : \mathcal{AO}_2^c[m] \rightarrow \bigsqcup_{k>0, m-kw_1^2 \geq 0} (\mathcal{AO}_2[m - kw_1^2] \times \mathcal{P}[k])$$

given by  $Y \mapsto (Y', \lambda^{(l)})$ . Then  $\overline{\Phi}[m]$  is an 1 – 1 and onto map with the preimage of  $(Y', \lambda') \in \mathcal{AO}_2[m - kw_1^2] \times \mathcal{P}[k]$  given by  $Y$  as follows:

$$Y := (\lambda'_r \cdot \epsilon^{-1}w_1^2)^{1+\lfloor \epsilon/2 \rfloor} \hookrightarrow (\lambda'_{r-1} \cdot \epsilon^{-1}w_1^2)^{1+\lfloor \epsilon/2 \rfloor} \hookrightarrow \dots (\lambda'_1 \cdot \epsilon^{-1}w_1^2)^{1+\lfloor \epsilon/2 \rfloor} \hookrightarrow Y',$$

where  $\lambda' = (\lambda'_1, \dots, \lambda'_{r-1}, \lambda'_r)$ . □

**Proof of Theorem 3.1** By definitions,  $\mathcal{AO}_1[t] = \mathcal{AO}_2[t]$  for  $0 \leq t \leq w_1^2 - \lfloor \epsilon/2 \rfloor$ . For the case of  $D_{n+1}^2$ ,  $\mathcal{AO}_1[w_1^2] \setminus \{(\epsilon^{-1}w_1^2, \epsilon^{-1}w_1^2, 0, \dots)\} = \mathcal{AO}_2[w_1^2] \setminus \{(w_1^2, 0, 0, \dots)\}$ . Thus  $|\mathcal{AO}_1[t]| = |\mathcal{AO}_2[t]|$  for  $0 \leq t \leq w_1^2$ . Proposition 3.4 tells us that for all  $m > w_1^2$ ,  $|\mathcal{AO}_i^c[m]|$  ( $i = 1, 2$ ) depend on the set of partition of  $\mathcal{AO}_i[l]$  and  $\mathcal{P}[k]$  such that  $k = \frac{m-l}{w_1^2} \in \mathbb{Z}_{>0}$ . Using an induction on  $m$ , we can conclude that  $|\mathcal{AO}_1^c[m]| = |\mathcal{AO}_2^c[m]|$ . Hence

$$|\mathcal{AO}_1[m]| = |\mathcal{AO}_2[m]|.$$

**Corollary 3.5.** *For the types of  $D_{n+1}^{(2)}$ ,  $A_{2n}^{(2)}$  and  $A_{2n-1}^{(2)}$ ,*

(1) *the generating functions of  $\mathcal{AO}_i$  ( $i = 1, 2$ ) are*

$$\chi_{\mathfrak{g}_n}^\Lambda(t) = \prod_{i=1}^{\infty} (1 + t^i)^{\kappa_i}, \quad \text{where } \begin{cases} \kappa_i = 0 & \text{if } \mathfrak{g}_n = A_{2n}^{(2)} \text{ and } i \equiv 0 \pmod{w_1^2}, \\ \kappa_i = 1 & \text{otherwise,} \end{cases}$$

$$(2) |\mathcal{Z}[m]| = \sum_{k \geq 0, m-kw_1^2 \geq 0} (|\mathcal{S}'[m - kw_1^2]| \times |\mathcal{P}[k]|) \text{ where } \begin{cases} \mathcal{S}' = \mathcal{S}_{(w_1^2)} & \text{if } \mathfrak{g}_n = A_{2n}^{(2)}, \\ \mathcal{S}' = \mathcal{S} & \text{otherwise.} \end{cases}$$

*Proof.* Note that  $\Delta$  in (1.1) coincides with  $w_1^2$ . By Proposition 2.4 and Remark 2.7(3), the assertions follow.  $\square$

Since the map  $P^t$  is injective, we can define the virtual characters of  $\mathcal{AO}_1$ . The following theorem tells that Theorem 3.1 for types of  $D_{n+1}^{(2)}$  and  $A_{2n}^{(2)}$  can be interpreted in more stronger sense.

**Theorem 3.6.** *For the types of  $D_{n+1}^{(2)}$  and  $A_{2n}^{(2)}$ , we have*

$$\chi_{\mathfrak{g}_n}^\Lambda = \overset{\circ}{\chi}_{\mathfrak{g}_n}(\mathcal{AO}_2).$$

*Proof.* By definition, the Young walls of those type do not contain the half-thickness block. Then we have  $\overset{\circ}{\chi}_{\mathfrak{g}_n}(Y[t]) = \overset{\circ}{\chi}_{\mathfrak{g}_n}(\mathcal{AO}_1[t])$  for  $0 \leq t \leq w_1^2$ . Using an induction, the assertion holds in a similar way of Theorem 3.1.  $\square$

**Proposition 3.7.** *For the types of  $B_n^{(1)}$ ,  $D_n^{(1)}$  and  $m \in \mathbb{Z}_{\geq 0}$ , we have bijections*

$$\Xi[m] : \mathcal{AO}_2[m] \rightarrow \bigsqcup_{k=0}^{\lfloor \frac{m}{w_1^1} \rfloor} (\mathcal{S}'[m - kw_1^1] \times \mathcal{S}[k]) \text{ where } \begin{cases} \mathcal{S}' = \mathcal{S}_{(w_1^2)} & \text{if } \mathfrak{g}_n = B_n^{(1)} \text{ and } \Lambda = \Lambda_n, \\ \mathcal{S}' = \mathcal{S} & \text{otherwise.} \end{cases}$$

*Proof.* Set  $\tilde{\epsilon} = 2$  if  $\mathfrak{g}_n = B_n^{(1)}$  and  $\Lambda \neq \Lambda_n$ , and  $\tilde{\epsilon} = 1$ , otherwise.

Let  $\Xi[m]$  be a map from  $\mathcal{AO}_2[m]$  to  $\bigsqcup_{k \geq 0, m - kw_1^1 \geq 0} \mathcal{S}'[m - kw_1^1]$  by the following algorithm (C):

- (C1) Let  $Y \in \mathcal{AO}_2[m]$  be given. Set  $Y^{(0)} = Y$ ,  $\lambda^{(0)} = (0)$ ,  $\hat{\lambda}_0 = 0$  and  $l = 0$ . If  $|Y| \in \mathcal{S}'[m]$ , define  $Y' = Y$  and terminate this algorithm.
- (C2) Find a maximal  $i$  such that

$$y_i^{(l)} = \hat{\lambda}_{l+1} \cdot \tilde{\epsilon}^{-1} w_1^1 \quad \text{or} \quad \hat{\lambda}_{l+1} \cdot \overline{\tilde{\epsilon}^{-1} w_1^1} \text{ for some } \hat{\lambda}_{l+1} \in 2\mathbb{Z}_{\geq 0} + 1,$$

and set  $t_{l+1}$  to be the number of parts in  $Y^{(l)}$  which are equal to  $y_i^{(l)}$ .

- (C3) Set  $a = \lfloor \frac{t_{l+1}}{\tilde{\epsilon}} \rfloor - c(y_i^{(l)})$  and define

$$Y^{(l+1)} := (y_1^{(l)}, y_2^{(l)}, \dots, y_{i-a}^{(l)}, y_{i+1}^{(l)}, \dots) \text{ and } \lambda^{(l+1)} = (\underbrace{\hat{\lambda}_{l+1}, \dots, \hat{\lambda}_{l+1}}_a) * \lambda^{(l)}.$$

- (C4) If  $Y^{(l+1)} \in \mathcal{S}'$ , then define  $Y' = Y^{(l+1)}$  and terminate this algorithm, otherwise  $l = l + 1$  and go to (C2).

Then this algorithm terminates in a finite step and the followings hold:

$$k := \frac{\Sigma_Y - \Sigma_{Y'}}{w_1^1} \in \mathbb{Z}_{\geq 0}, \quad Y' \in \mathcal{S}'[m - kw_1^1], \quad \lambda := \lambda^{(l)} \in \mathcal{O}[k].$$

Thus we can get a function

$$\Xi[m] : \mathcal{AO}_2[m] \rightarrow \bigsqcup_{k \geq 0, m - kw_1^1 \geq 0} (\mathcal{S}'[m - kw_1^1] \times \mathcal{O}[k])$$

given by  $Y \mapsto (Y', \lambda)$ . Then  $\Xi[m]$  is an 1-1 and onto map with the preimage of  $(Y', \lambda') \in \mathcal{S}'[m - kw_1^1] \times \mathcal{O}[k]$  given by  $Y$  as follows:

$$Y := (\lambda'_r \cdot \tilde{\epsilon}^{-1} w_1^1)^{1 + \lfloor \tilde{\epsilon}/2 \rfloor} \hookrightarrow (\lambda'_{r-1} \cdot \tilde{\epsilon}^{-1} w_1^1)^{1 + \lfloor \tilde{\epsilon}/2 \rfloor} \hookrightarrow \dots \hookrightarrow (\lambda'_1 \cdot \tilde{\epsilon}^{-1} w_1^1)^{1 + \lfloor \tilde{\epsilon}/2 \rfloor} \hookrightarrow Y'.$$

Thus, by the Sylvester's bijection, we have a bijection

$$\mathcal{AO}_2[m] \rightarrow \bigsqcup_{k=0}^{\lfloor \frac{m}{w_1^1} \rfloor} (\mathcal{S}'[m - kw_1^1] \times \mathcal{S}[k]).$$

□

By a similar way as Corollary 3.5, we have the following lemma from Proposition 3.7:

**Corollary 3.8.** *For the types of  $B_n^{(1)}$  and  $D_n^{(1)}$ ,*

(1) *the generating functions of  $\mathcal{AO}_i$  ( $i = 1, 2$ ) are*

$$\chi_{\mathfrak{g}_n}^\Lambda(t) = \prod_{i=1}^{\infty} (1 + t^i)^{\kappa_i}, \text{ where } \begin{cases} \kappa_i = 2 & \begin{array}{l} \text{if } \mathfrak{g}_n = B_n^{(1)}, \Lambda \neq \Lambda_n \text{ and } i \equiv 0 \pmod{w_1^2}, \\ \text{or if } \mathfrak{g}_n = B_n^{(1)}, \Lambda = \Lambda_n \text{ and } i \equiv w_1^1 \pmod{w_1^2}, \\ \text{or if } \mathfrak{g}_n = D_n^{(1)} \text{ and } i \equiv 0 \pmod{w_1^1}, \end{array} \\ \kappa_i = 1 & \text{otherwise,} \end{cases}$$

$$(2) |\mathcal{Z}[m]| = \sum_{\substack{l \geq 0, \\ m - lw_1^2 \geq 0}} \left\{ \left( \sum_{\substack{k \geq 0, \\ m - lw_1^2 - kw_1^1 \geq 0}} (|\mathcal{S}'[m - lw_1^2 - kw_1^1] \times \mathcal{S}[k]|) \right) \times \mathcal{P}[l] \right\}.$$

Since we consider only the parts which can be repeated in  $\mathcal{Z}$ , we have an identity more general than the one in Theorem 3.1 as follows:

**Corollary 3.9.** *Let  $X = \{x_1, \dots, x_r\} \in \mathbb{N}^r$  with  $1 \leq x_1 < \dots < x_r < w_3^1$  and  $\mathcal{AO}_i^X$  be a subset of  $\mathcal{AO}_i$  ( $i = 1, 2$ ) satisfying the following condition:*

*Each part is not congruent to  $x_j$  modulo  $w_3^1$  for all  $1 \leq j \leq r$ .*

*Then we have*

$$|\mathcal{AO}_1^X[m]| = |\mathcal{AO}_2^X[m]| \quad \text{for all } m \in \mathbb{Z}_{\geq 0}.$$

#### 4. BESSENRODT'S REFINEMENT

In the previous section, we show that  $|\mathcal{AO}_1[m]|$  (or  $|\mathcal{AO}_1^X[m]|$ ) coincides with  $|\mathcal{AO}_2[m]|$  (or  $|\mathcal{AO}_2^X[m]|$ ). In this section, we will construct an explicit bijection between two sets, by generalizing the technique of Bessenrodt's insertion algorithm [1, 2, 4, 26]. Recall the fact that, in the case of  $D_{n+1}^{(2)}$  and  $A_{2n}^{(2)}$ , the map  $P$  is injective. Thus we can apply and use the "partition identities" very efficiently.

This section is devoted to prove the following theorem:

**Theorem 4.1.** *For all  $m \in \mathbb{Z}_{>0}$ , there is a bijection*

$$\Theta : \mathcal{AO}_1[m] \rightarrow \mathcal{AO}_2[m].$$

**4.1. Types of  $D_{n+1}^{(2)}$  and  $A_{2n}^{(2)}$ .**

**4.1.1.  $A_{2n}^{(2)}$ .** Let  $N \in \mathbb{Z}_{>0}$  and let  $X_N = \{x_1, x_2, \dots, x_r\}$  with  $1 \leq x_1 < x_2 < \dots < x_r < N$ .

**Definition 4.2.** Let  $\mathcal{AO}_1^{X_N}[m]$  denote the set of partitions of  $m$  satisfying the following conditions:

- (1) Each part is congruent to 0 or some  $x_i$  modulo  $N$ .
- (2) Only the multiples of  $N$  can be repeated and the smallest part is less than  $N$ .

- (3) The difference between two successive parts is at most  $N$  and strictly less than  $N$  if either part is divisible by  $N$ .

**Definition 4.3.** Let  $\mathcal{AO}_2^{X_N}[m]$  denote the set of partitions of  $m$  satisfying the following conditions:

- (1) Each part is congruent to some  $x_i$  modulo  $N$ .
- (2) No part can be repeated.

**Theorem 4.4.** [1, 2] Let  $m \in \mathbb{Z}_{\geq 0}$ .

- (1)  $|\mathcal{AO}_1^{X_N}[m]| = |\mathcal{AO}_2^{X_N}[m]|$ .
- (2) There is an insertion algorithm which establishes an explicit bijection

$$\Theta : \mathcal{AO}_1^{X_N}[m] \rightarrow \mathcal{AO}_2^{X_N}[m].$$

Since we will use and generalize the insertion algorithm in later subsections, we don't give an overview of this algorithm.

**Proof of Theorem 4.1** ( $A_{2n}^{(2)}$ ) If we set  $X_{w_1^2} = \{1, 2, 3, \dots, w_1^2 - 1\}$ , one can easily check that

$$\mathcal{AO}_i[m] \text{ coincides with } \mathcal{AO}_i^{X_{w_1^2}}[m] \quad (i = 1, 2).$$

Thus we have bijection between  $\mathcal{AO}_1$  and  $\mathcal{AO}_2$ . Moreover, the bijection in Theorem 4.4 is a weight-preserving map. [2]

4.1.2.  $D_{n+1}^{(2)}$ .

**Definition 4.5.** Let  $\mathcal{AO}_3^{X_{2N}}[m]$  denote the set of partitions of  $m$  satisfying the following conditions:

- (1) Each part is congruent to 0 or some  $x_i$  modulo  $2N$ .
- (2) Only the multiples of  $N$  can be repeated and the smallest part is less than  $2N$ .
- (3) The difference between two successive parts is at most  $2N$  and strictly less than  $2N$  if either part is divisible by  $N$ .

**Definition 4.6.** Let  $\mathcal{AO}_4^{X_{2N}}[m]$  denote the set of partitions of  $m$  satisfying the following conditions:

- (1) Each part is congruent to some  $x_i$  modulo  $2N$ .
- (2) Only the multiples of  $N$  can be repeated.

**Theorem 4.7.** [26] For any  $m \in \mathbb{Z}_{\geq 0}$ , there is an insertion algorithm which establishes an explicit bijection

$$\Theta' : \mathcal{AO}_3^{W_{2N}}[m] \leftrightarrow \mathcal{AO}_4^{W_{2N}}[m].$$

**Proof of Theorem 4.1** ( $D_{n+1}^{(2)}$ ) If we set  $X_{w_1^2} = \{1, 2, \dots, w_1^2 - 1\}$ , one can easily check that

$$\mathcal{AO}_1[m] = \mathcal{AO}_3^{X_{w_1^2}}[m], \quad \text{for all } m \in \mathbb{Z}_{\geq 0}.$$

Thus it suffices to show that there is a bijection between  $\mathcal{AO}_4^{X_{w_1^2}}[m]$  and  $\mathcal{AO}_2[m]$ . This bijection can be constructed by using the Sylvester's bijection [20] between  $\mathcal{O}$  and  $\mathcal{S}$ .

We can extract an odd partition  $\lambda$  from  $\mu \in \mathcal{AO}_4^{X_{w_1^2}}[m]$  by the following algorithm (D):

- (D1) Let  $\mu \in \mathcal{AO}_4^{X_{w_1^2}}[m]$ . Set  $\mu^{(0)} = \mu$  and  $l = 0$ .
- (D2) Find a maximal  $i$  such that  $\mu_i^{(l)} = \lambda_{l+1} w_1^2$  for some  $\lambda_{l+1} \in 2\mathbb{Z}_{\geq 0} + 1$ , and set

$$\mu^{(l+1)} = (\mu_1^{(l)}, \mu_2^{(l)}, \dots, \mu_{i-1}^{(l)}, \mu_{i+1}^{(l)}, \dots).$$

(D3) If there is no  $j$  such that

$$\mu_j^{(l+1)} = kw_2^2 \text{ for some } k \in 2\mathbb{Z}_{\geq 0} + 1,$$

define  $\underline{\mu} = \mu^{(l+1)}$  and terminate this algorithm, otherwise,  $l = l + 1$  and go to (D2).

This algorithm terminates in a finite step and  $\lambda := (\lambda_l, \lambda_{l-1}, \dots, \lambda_1)$  is an odd partition.

Then by the Sylvester's bijection, we have a strict partition  $\lambda'$  and we have

$$\mu' := (\lambda'_r w_2^2) \hookrightarrow (\lambda'_{r-1} w_2^2) \hookrightarrow \dots \hookrightarrow (\lambda'_1 w_2^2) \hookrightarrow \underline{\mu} \in \mathcal{AO}_2[m].$$

In conclusion, we can construct a bijection  $\mathcal{AO}_1 \rightarrow \mathcal{AO}_2$  which preserves weight.

**Definition 4.8.** Let  $\mathcal{AO}_5^{X_{2N}}[m]$  denote the set of partitions of  $m$  satisfying the following conditions:

- (1) Each part is congruent to some  $x_i$  modulo  $2N$  and to 0 modulo  $2N$  if  $N \in X_{2N}$ .
- (2) No part can be repeated.

**Corollary 4.9.** For any  $m \in \mathbb{Z}_{\geq 0}$ , there is an insertion algorithm which establishes an explicit bijection

$$\Theta'' : \mathcal{AO}_4^{X_{2N}}[m] \rightarrow \mathcal{AO}_5^{X_{2N}}[m],$$

and hence

- (1) there exists a bijection  $\Theta'' \circ \Theta' : \mathcal{AO}_3^{X_{2N}}[m] \rightarrow \mathcal{AO}_5^{X_{2N}}[m]$ ,
- (2)  $|\mathcal{AO}_3^{X_{2N}}[m]| = |\mathcal{AO}_4^{X_{2N}}[m]| = |\mathcal{AO}_5^{X_{2N}}[m]|$ .

**4.2. For the other types.** For  $\mathfrak{g}_n = B_n^{(1)}$  and  $\Lambda = \Lambda_n$ , set  $\mathcal{AO}_{1^\circ} := \mathcal{AO}_1$  and  $\mathcal{AO}_{2^s} := \mathcal{AO}_2$ .

**Proposition 4.10.** Let  $\mathcal{AO}_{1 \cap 2^s} := \mathcal{AO}_1 \cap \mathcal{AO}_{2^s}$ . For all  $m \in \mathbb{Z}_{>0}$ , there are injective maps

$$\overline{\Theta}[m] : \mathcal{AO}_{2^s}[m] \rightarrow \bigsqcup_{k \geq 0, m - kw_1^1 \geq 0} (\mathcal{AO}_{1 \cap 2^s}[m - kw_1^1] \times \mathcal{P}[k])$$

given by

$$Y \mapsto (Y', \lambda) \quad \text{such that} \quad \ell(\lambda) \leq \ell(Y') = \ell(Y).$$

*Proof.* Let  $\Theta[m]$  be a map from  $\mathcal{AO}_{2^s}[m]$  to  $\bigsqcup_{k \geq 0, m - kw_1^2 \geq 0} \mathcal{AO}_{1 \cap 2^s}[m - kw_1^2]$  by following algorithm (D):

- (D1) Let  $Y \in \mathcal{AO}_{2^s}[m]$  and set  $Y^{(0)} = Y$  and  $l = 0$ . If  $Y \in \mathcal{AO}_{1 \cap 2^s}[m]$ , then terminate this algorithm and set  $Y' = Y$ .
- (D2) Find a maximal  $i$  such that

$$\begin{cases} y_{i-1}^{(l)} - y_i^{(l)} \succ w_1^2 & \text{if } \mathfrak{g}_n = A_{2n-1}^{(2)}, \\ y_{i-1}^{(l)} - y_i^{(l)} \succ w_1^2, \text{ or } y_{i-1}^{(l)} - y_i^{(l)} = w_1^2 \text{ and } y_{i-1}^{(l)} \equiv w_3^1 \pmod{w_1^2} & \text{if } \mathfrak{g}_n = B_n^{(1)} \text{ and } \Lambda \neq \Lambda_n, \\ y_{i-1}^{(l)} - y_i^{(l)} \succ w_1^2, \text{ or } y_{i-1}^{(l)} - y_i^{(l)} = w_1^2 \text{ and } y_{i-1}^{(l)} \equiv w_3^1 \text{ or } \overline{w_3^1} \pmod{w_1^2} & \text{otherwise.} \end{cases}$$

Then set

$$Y^{(l+1)} := (y_1^{(l)} - w_1^2, y_2^{(l)} - w_1^2, \dots, y_{i-1}^{(l)} - w_1^2, y_i^{(l)}, y_{i+1}^{(l)}, \dots).$$

- (D3) If  $Y^{(l+1)} \in \mathcal{AO}_{1 \cap 2^s}$ , then  $Y' = Y^{(l+1)}$  and terminate this algorithm, otherwise  $l = l + 1$  and go to (D2).

Then this algorithm terminates in a finite step and the followings hold:

- $k := \frac{\Sigma_Y - \Sigma_{Y'}}{w_1^2} \in \mathbb{Z}_{\geq 0}$ ,  $Y' \in \mathcal{AO}_{1 \cap 2^s}[m - kw_1^2]$ ,



Then the way of obtaining  $Y'$  and  $\lambda$  from  $Y$  can be interpreted by the following algorithm **(E)**:

- (Ei)** ( $1 \leq i \leq t$ ) First, remove columns of  $i \cdot w_1^2$  for  $l_i - (\varepsilon_i - \varepsilon_{i-1})$  times. Then also remove "L"-hook of  $w_1^2$  whose length is  $i$  if there are any. This case occurs only if  $j + (t - r_j - 1) = i$  and  $\varepsilon_j - \varepsilon_{j-1} = 1$  for some  $j < i$ . Set the obtained Young wall  $Y^{(i)}$ .

Then this algorithm terminates at  $t$ -th step and  $Y^{(t)} = Y' \in \mathcal{AO}_{1^o \cap 2}[\Sigma_Y - \Sigma_\lambda w_1^2]$ .

For a  $k \in \mathbb{Z}_{\geq 0}$ , define the right inserting  $(k \cdot w_1^2)^j$  into  $\lambda \in \mathcal{AO}_{1^o \cap 2}$  denoted by  $\lambda \leftarrow (k \cdot w_1^2)^j$  as follows:

$$\lambda \leftarrow (k \cdot w_1^2)^j := (\lambda_1, \lambda_2, \dots, \lambda_i, \underbrace{k \cdot w_1^2, \dots, k \cdot w_1^2}_j, \lambda_{i+1}, \dots) \quad \text{for } \lambda_i \succeq k \cdot \overline{w_1^2} \succ \lambda_{i+1}.$$

From now, we will give a reverse algorithm which is a kind of insertion algorithm. Assume we have  $\triangleright \lambda = (1^{\mu_1}, 2^{\mu_2}, \dots, t^{\mu_t})$  and  $Y^{(t)} \in \mathcal{AO}_{1^o \cap 2}$ .

The way of obtaining  $Y$  from  $Y' = Y^{(t)}$  can be interpreted as the following algorithm **(F)**:

We start from  $i = t$  to  $i = 1$ .

- (Fi)** If  $Y^{(i)} \leftarrow (i \cdot w_1^2)^{\mu_i} \in \mathcal{AO}_{1^o}$ , set  $Y^{(i-1)} := Y^{(i)} \leftarrow (i \cdot w_1^2)^{\mu_i}$ . Otherwise, insert as many "L"-hooks of  $w_1^2$  of length  $i$  as necessary and then insert columns of  $i \cdot w_1^2$  until the number of columns and hooks together is  $\mu_i$ . Set the resulting Young wall  $Y^{(i-1)}$ .

Then this insertion algorithm always works,  $Y^{(i)} \in \mathcal{AO}_{1^o}$  for all  $0 \leq i < t$  and  $Y = Y^{(1)}$ .  $\square$

**Proof of Theorem 4.1** For  $t \in \mathbb{Z}_{\geq 0}$ , there is a bijection

$$\{\lambda \in \mathcal{P} \mid \ell(\lambda) \leq t\} \rightarrow \{\lambda \in \mathcal{P} \mid \lambda_1 \leq t\} \quad \text{given by} \quad \lambda \mapsto \lambda^{\text{tr}}.$$

Thus, for  $m \in \mathbb{Z}_{\geq 0}$ , we have a bijection

$$\mathcal{AO}_1[m] \longrightarrow \mathcal{AO}_2[m]$$

given by

$$Y_1 \xrightarrow{P^o} Y_2 \xrightarrow{\text{(E)}} (Y'_2, \lambda) \xrightarrow{(P^s, \text{tr})} (Y'_3 := P^s(Y'_2), \lambda^{\text{tr}}) \xrightarrow{Y'_3 + w_1^2 \cdot \lambda^{\text{tr}}} Y_3 \xrightarrow{P^{s-1}} Y_4.$$

To express the algorithm given in Proposition 4.11 more precisely, we will give the following example.

**Example 4.13.** In this example, we assume that  $\mathfrak{g}_n = A_7^{(2)}$  and keep the notation given above.

- (1) Let  $Y_1 = (33, 31, \overline{28}^2, 21^2, 15, 9, 7, 1)$  be a partition in  $\mathcal{AO}_1[194]$ . Then we have

$$Y_2 := P^o(Y_1) = (33, 31, \overline{28}, 28, \overline{21}, 21, 15, 9, 7, 1) \quad \text{and} \quad Y'_2 = (26, 24, \overline{21}, \overline{14}, 8, 2, 1).$$

To get the parameters  $l_i$ ,  $\varepsilon_i$  and  $r_i$  appearing in (4.1), we write  $\triangleright Y_2$  as follows:

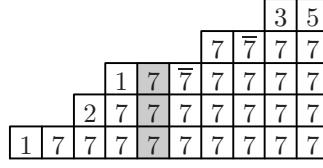
$$\triangleright Y_2 = (1, 7, (2+7), (8+7), 21, (\overline{14}+7), 28, (\overline{21}+7), (24+7), (26+7)).$$

Hence we obtain

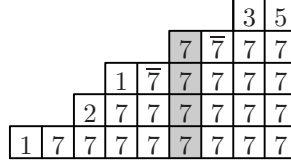
$i$	0	1	2	3	4	5	6	7
$l_i$	—	1	0	1	1	0	0	0
$\varepsilon_i$	0	1	1	1	1	0	0	0
$r_i$	1	2	3	4	7	0	0	0

Thus, by the formula (4.1),  $\triangleright \lambda = (3, 4, 7)$ . Now we start the combinatorial algorithm **(E)** with  $Y$ . The 7-modular Young diagram for  $Y_2 = Y^{(0)}$  is

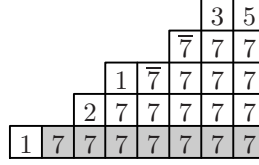




Here  $\mu_1 = 0$ , since we cannot remove the column  $\boxed{7}$  without violating the  $\mathcal{AO}_{1^0}$ -condition. Since there is no column  $\boxed{7 \ 7}$  and also no L-hook of length 2, we also have  $\mu_2 = 0$  and  $Y^{(0)} = Y^{(1)} = Y^{(2)}$ . Then we can remove one column  $\boxed{7 \ 7 \ 7}$ , but there is no hook of length 3 to be removed. So  $\mu_3 = 1$  and  $Y^{(3)}$  is



At this step, we can remove the column  $\boxed{7 \ 7 \ 7 \ 7}$  and there is no extra hook of length 4. Thus  $\mu_4 = 1$  and  $Y^{(4)}$  is



There is neither a column nor an admissible  $L$ -hook of length 5 and 6, so  $\mu_5 = \mu_6 = 0$  and  $Y^{(4)} = Y^{(5)} = Y^{(6)}$ . At seventh step, there is no column of length 6 but there is an admissible  $L$ -hook of length 7 which is shaded. Thus  $\mu_7 = 1$  and  $Y'_2 = Y^{(7)} = (26, 24, \overline{21}, \overline{14}, 8, 2, 1)$  is the desired one.

- (2) Let  $Y_1 = (33, 31, 28^2, \overline{21}^2, 15, 9, 7, 1)$  be a partition in  $\mathcal{AO}_1[194]$ . Then we have

$$Y_2 := P^o(Y_1) = (33, 31, 28, 28, \overline{21}, 21, 15, 9, 7, 1) \quad \text{and} \quad Y'_2 = (19, 17, \overline{14}, 9, 2, 1).$$

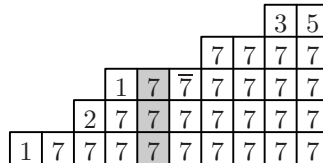
To get the parameters  $l_i$ ,  $\varepsilon_i$  and  $r_i$  appearing in (4.1), we write  $\triangleright Y_2$  as follows:

$$\triangleright Y_2 = (1, 7, (2+7), (8+7), 21, (\overline{14}+7), 28^2, (17+22 \cdot 7), (19+2 \cdot 7)).$$

Hence we obtain

$$\begin{array}{rcccccccc} i & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ l_i & - & 1 & 0 & 1 & 2 & 0 & 0 \\ \varepsilon_i & 0 & 1 & 1 & 1 & 2 & 0 & 0 \\ r_i & 1 & 2 & 3 & 4 & 6 & 0 & 0 \end{array}$$

Thus, by the (4.1),  $\triangleright \lambda = (3, 4, 6^2)$ . Now we start the combinatorial algorithm **(E)** with  $Y$ . The 7-modular Young diagram for  $Y_2 = Y^{(1)}$  is



By the same reason as in (1), we have  $\mu_1 = \mu_2 = 0$ ,  $Y^{(1)} = Y^{(2)}$  and  $\mu_3 = 1$ . Then  $Y^{(3)}$  is

							3	5
					7	7	7	7
		1	$\bar{7}$	7	7	7	7	7
	2	7	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7

At this step, we can remove the column  $\boxed{77777}$ , and again there is no extra hook of length 4. Thus  $\mu_4 = 1$  and  $Y^{(4)}$  is

							3	5
					7	7	7	7
		1	$\bar{7}$	7	7	7	7	7
	2	7	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7

There is neither a column nor an admissible  $L$ -hook of length 5  $\mu_5 = 0$  and  $Y^{(4)} = Y^{(5)}$ . However, there is one  $L$ -hook of length 6 which is shaded. After we remove the  $L$ -hook, we have another  $L$ -hook of length 6 by the following:

							3	5
					1	$\bar{7}$	7	7
		2	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7

Thus we have  $\mu_6 = 2$  and  $Y'_2 = Y^{(6)} = (19, 17, \bar{14}, 8, 2, 1)$  is the desired one.

- (3) For given  $Y'_2 = (19, 17, 14_F, 9, 2, 1) \in \mathcal{AO}_{1 \circ \cap 2}[61]$  and  $\triangleright \lambda = (3, 4, 6^2) \vdash 19$ , we will get a  $Y_2 \in \mathcal{AO}_{1 \circ}[194]$  by the following way.  $Y^{(6)} = Y'_2$  can be depicted by follows:

							3	5
					1	$\bar{7}$	7	7
	1	2	7	7	7	7	7	7

Then there is only one  $L$ -hook of length 6. Thus we have

							3	5
					1	$\bar{7}$	7	7
		2	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7

At this step, there seem to be two choices of  $L$ -hook of length 6:

							3	5
					1	$\bar{7}$	7	7
		2	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7

							3	5
					1	$\bar{7}$	7	7
		2	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7

After inserting the  $L$ -hook of length 6 to both of them, we have

							3	5
					$\bar{7}$	7	7	7
		1	7	7	7	7	7	7
	2	7	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7

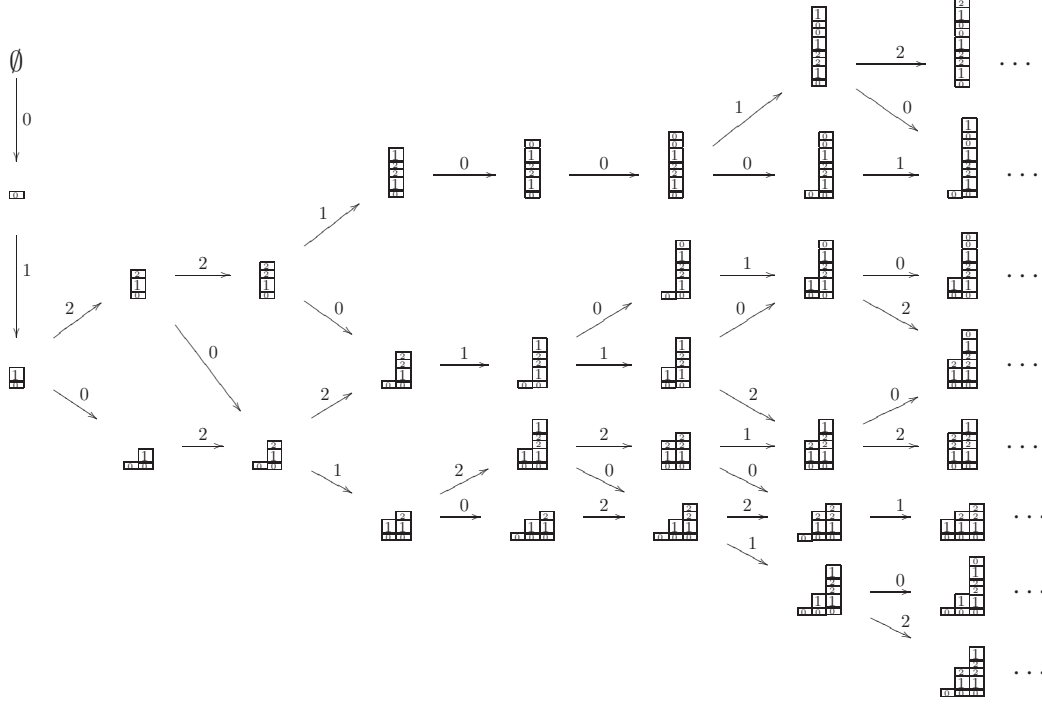
							3	5
					1	$\bar{7}$	7	7
		2	7	7	7	7	7	7
1	7	7	7	7	7	7	7	7

But the first one is not contained in  $\mathcal{AO}_{1 \circ}$ . Thus there is only one choice actually. Hence  $Y^{(5)}$  is the second one. Since the column  $\boxed{77777}$  can be inserted, so  $Y^{(3)}$  is

$$\text{wt}(\lambda) := \text{wt}(Y) \quad \text{for } \Theta(Y) = \lambda, Y \in Y(\Lambda).$$

## APPENDIX A. PATTERNS OF YOUNG WALLS

[illegible]

APPENDIX B. THE CRYSTAL STRUCTURE ON  $\mathcal{S}$  OF TYPE  $D_3^{(2)}$ 

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